Diversification in the Presence of Taxes

There are substantial risks incurred with concentrated holdings.

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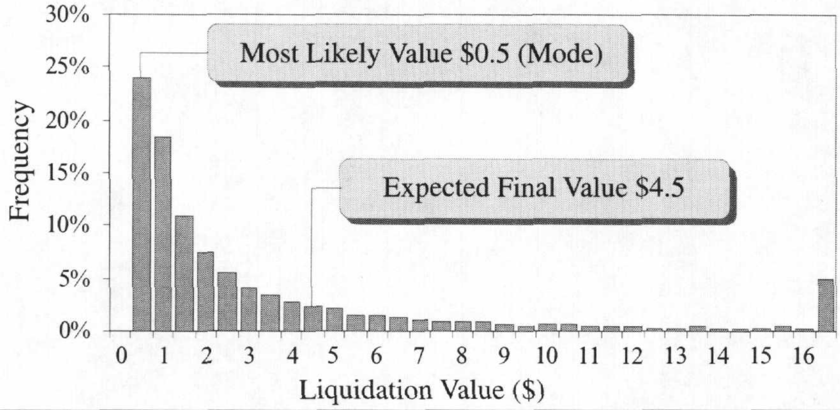
Taxes can have a considerable impact on portfolio value, and investment decisions should be made with a clear understanding of the tax-adjusted performance of the alternatives under consideration. The impact of taxes has received increased attention in recent years with regard to portfolio performance (Stein [1998]); asset allocation (Jacob [1995]); manager selection (Jeffrey and Arnott [1993]); and tax efficiency (Dickson and Shoven [1993]).

One common—and key—problem that taxable investors face is diversifying a low cost basis single-asset or concentrated portfolio. In the tax-exempt case, modern portfolio theory is very clear on the benefits of diversification, and there have evolved useful industry standard methods for addressing this, such as the mean-variance approach to optimal portfolio diversification (Markowitz [1987]). In the presence of taxes, however, there are no standard approaches for arriving at a considered choice.

Taxes complicate the analysis because capital gains taxes are incurred at the time of diversification. The tax resulting from the sale of a portion of the initial asset reduces the possibility of future returns, and may or may not outweigh any uncertain future benefit from diversification.

In proposing an approach to solving the taxable investor's diversification dilemma, we consider a very much simplified problem in which there are just two possible assets: the initial holdings and a diversified benchmark portfolio. Our framework considers the investment

EXHIBIT 1
AFTER-TAX HORIZON LIQUIDATION VALUE—
INITIAL $1 CONCENTRATED SECURITY WITH VOLATILITY 40%



Expected Final Value	$4.5
Standard Deviation	$13.0
Max. Likelihood	$0.5
Probability of $1 or less	43%
Probability of $2 or less	60%

decision *with* initial taxes, and shows how it can be viewed as an equivalent but much simpler investment decision *without* initial taxes. We do this by creating a *tax-deferred* investor (with different investment opportunities) who does not pay initial capital gains taxes, but whose final investment performance is identical to that of the actual investor.[1]

Identifying the best diversification decision available to the tax-deferred investor leads to a decision for the actual investor. The tax-deferred investor is in a sense a tax-deferred equivalent of the actual investor, facing an equivalent but simpler problem.

We present results to the problem under specific numerical assumptions, and investigate the sensitivity of the results to the key parameters, namely, the risk of the initial holding, the investment horizon, the cost basis, excess expected return, and riskless rate. In an example, we consider diversifying a portfolio so as to reduce *tracking error* risk using a mean-variance optimizer. An appendix provides technical details.

RISKS AND BENEFITS OF DIVERSIFICATION: AN EXAMPLE

We assume initially an investor with an initial high-risk holding who owns $1 million concentrated in a risky stock with a zero cost basis. We compare the distributions of uncertain end-of-horizon future values without and with diversification, and contrast these with the much simpler comparison (of yearly expected return and risk) that may be used to decide the level of diversification for a tax-deferred investor.

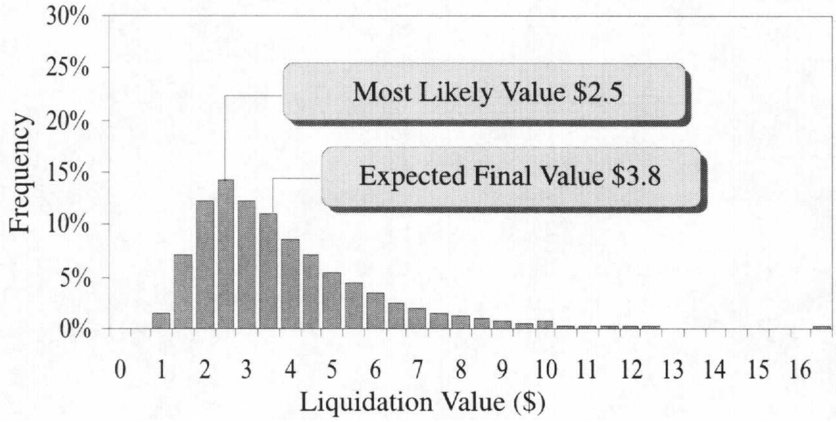
First consider the initial undiversified holding. Let us assume that the risk is 40%, measuring risk as the annual standard deviation (volatility) of rate of return, and that the investment horizon is 20 years. Let us further assume that the stock returns an expected 10% per year, of which 7% is price appreciation and 3% dividend yield. Each year, dividends are taxed at 39.6%, and the after-tax dividend proceeds are reinvested in the portfolio. We also assume that the investor liquidates the holdings and incurs capital gains taxes of 20% at the horizon.

The investor's final wealth is uncertain, and Exhibit 1 shows its distribution obtained from a Monte Carlo simulation in which the security prices follow a lognormal process. While the investor can expect $1 million to grow on average to $4.5 million after taxes, the distribution of final wealth is unattractively broad. The mode (most likely value) of the distribution is only $0.5 million; the probability of ending up with less than the initial $1 million is 43%; and the probability of not keeping up with inflation is 60%.[2]

Next, consider the consequences of a decision to diversify. Suppose that the investor liquidates the risky holding, paying taxes at the 20% rate, and invests the remaining $800,000 in a diversified portfolio with an annual standard deviation of 15%, but with the same expected price and dividend returns as the risky stock. After 20 years, tax is paid at the 20% rate (reduced by a cost basis of $800,000).

Exhibit 2 shows the final wealth distribution. The investor can expect to have only $3.8 million, on average, after 20 years. While this expectation is lower than the value in Exhibit 1 because only 80% of the initial

EXHIBIT 2
HORIZON AFTER-TAX LIQUIDATION VALUE—INITIAL $0.80 DIVERSIFIED PORTFOLIO WITH VOLATILITY 15%



Expected Final Value	$3.8
Standard Deviation	$2.4
Max. Likelihood	$2.5
Probability of $1 or less	2%
Probability of $2 or less	21%

EXHIBIT 3
TRADE-OFF BETWEEN EXPECTED RETURN AND RISK IN ABSENCE OF TAXES

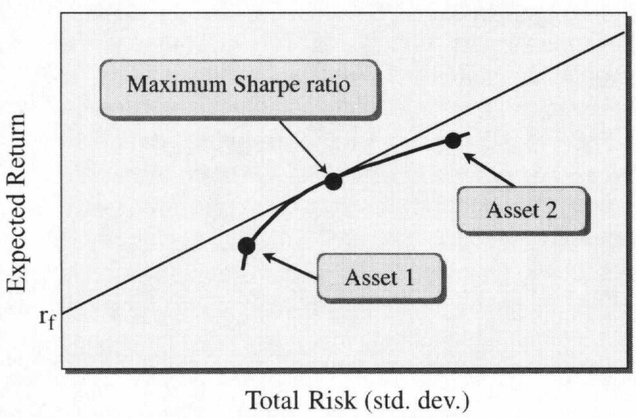


EXHIBIT 4
RISK OF SELECTED INVESTMENTS 1994-1999

S&P 500	14%
Exxon	16%
GE	21%
IBM	30%
Microsoft	35%
Micron Technologies	72%
AOL*	84%
Amazon.com*	124%

Risk is measured by annualized standard deviation.

*Based on 1997-1999.

value is available for compounding, the probability distribution is nonetheless more attractive. The mode increases from $0.5 million to $2.5 million; the chance of ending up with less than the initial $1 million drops from 43% to just 2%; and the chance of not keeping up with inflation falls from 60% to 21%.

The graphs show that diversification boosts performance when taxes are incurred even though the asset expected returns are identical. How can this be? Diversification makes the expected return more readily achievable. In Exhibit 1, with a high volatility, the expected return is in the "tail" of the distribution and is not likely to be achieved; in Exhibit 2, with a lower volatility, the expected return is more likely to be achieved.[3]

Comparing Exhibits 1 and 2, we ask whether the initial tax is justified. How should the investor trade off risk (lessened by diversification) against anticipated future wealth (also lessened by initial taxes)? In particular, how much of the initial holding should be diversified, and by what principle can this be justified? Our approach provides a solution that is both analytical and intuitive.

The approach is based on the fact that the tax-exempt diversification problem is simpler than that faced by an investor who must pay initial taxes in order to diversify. Exhibit 3 shows the well-known tax-exempt trade-off between risk and return.[4] The Sharpe ratio criterion is a common one for selecting a diversification level; the investor chooses the fraction of initial asset (Asset 2) to sell (purchasing shares of Asset 1) so that the resulting portfolio has the highest ratio of excess return (above the risk-free rate r_f) to standard deviation of excess return.

In order to place the 40% standard deviation example of Exhibit 1 in context, Exhibit 4 shows recent standard deviation risks of some well-known stocks. The 40% volatility of Exhibit 1, while higher than that of many large-cap securities, is not particularly high for a technology or small company. Our base benchmark volatility of 15% is similar to the volatility of the S&P 500 index over a recent five-year period.

AN APPROACH TO THE PROBLEM

We now specify an analytical framework for the opportunities and choices of the actual investor and the tax-deferred investor, and indicate how to match them so as to have very nearly the same future cash flows. This matching allows us to view the diversification decision in the presence of taxes (faced by the actual investor) as a conventional risk-return trade-off without initial tax complications (faced by the tax-deferred investor). We use the maximum Sharpe ratio as the decision criterion for making the optimal diversification choice of the tax-deferred investor. If this is the best choice for the tax-deferred investor, and the actual investor's future cash flows closely match, it follows that we have also identified an optimal choice for the actual investor.

In the simplified problem, the investor holds an initial portfolio, A, with market value W_0, which may be sold in whole or in part to purchase shares in a fully diversified benchmark portfolio, B. The investor's goal is to select the best fraction x (between 0 and 1) of the initial portfolio to be sold, and invest the after-tax proceeds in B. The resulting position is then held for the preset investment horizon of n years, during which time uncertain rates of return for the initial asset and the benchmark are observed and compounded. At the end of the investment horizon, the position is liquidated and taxes are paid. This defines a probability distribution of final after-tax values for the actual investor to which we will match a tax-deferred investor.[5]

Imagine, now, a tax-deferred investor who can diversify without initially paying capital gains taxes, but who pays taxes on liquidation and whose after-tax horizon investment performance matches that of the actual investor very closely. The tax-deferred investor must face different investment opportunities; in particular, the tax-deferred assets must pay a lower expected rate of return to compensate for taxes paid by the actual investor. The tax-deferred investor holds an initial portfolio A^* (with market value W_0), sells a fraction x^*, and (without paying initial taxes or changing the cost basis) uses the proceeds to purchase shares of the benchmark, B^*. The resulting portfolio is held for n years, and the position is then liquidated and taxes are paid. This defines a probability distribution of final after-tax values for the tax-deferred investor.[6]

Inputs to the model are as follows. The expected rates of return for A and B are μ_A and μ_B. Their annual standard deviations of return are σ_A and σ_B, and the beta of A with respect to the benchmark B is β. The horizon is fixed at n years, after which the investment position is liquidated. The tax rate on long-term capital gains is τ; there are no dividends; and the risk-free rate is r_f.

The mathematical formulation and its solution are in the appendix, where we derive the tax-deferred investor's rate of return μ_x and standard deviation σ_x analytically in terms of our inputs for each value of the diversification fraction x under the assumption of joint lognormal asset returns. To match the final cash flow distributions of the actual and tax-deferred investors, we choose x^* to match diversification exposure, and set the joint performance of A^* and B^* to compensate for taxes paid by the actual investor.

THE DIVERSIFICATION SOLUTION: EXAMPLE

We provide an example to study the actual investor's diversification decision, as chosen by applying the maximum Sharpe [1964] ratio criterion to the tax-deferred investor. We then use sensitivity analysis to show that greater diversification is associated with: greater initial asset volatility, longer investment horizon, higher cost basis, lower expected return of the initial asset, and a lower risk-free rate. Less diversification is needed when the investor receives a step-up in basis at the horizon.

As our base case, we set numerical values for the initial asset A and the benchmark B as follows:

Expected returns:	$\mu_A = \mu_B = 10\%$
Volatilities:	$\sigma_A = 25\%$, $\mu_B = 15\%$
Horizon:	$n = 20$ years
Tax rate:	$\tau = 20\%$ on capital gains
Risk-free rate:	$r_f = 6\%$
Initial cost basis	$C_0 = 0$

Exhibit 5 shows the after-tax annual expected return and risk trade-off faced by the tax-deferred investor in a representation analogous to Exhibit 3. In this case,

EXHIBIT 5
ACTUAL INVESTOR'S DIVERSIFICATION DECISION

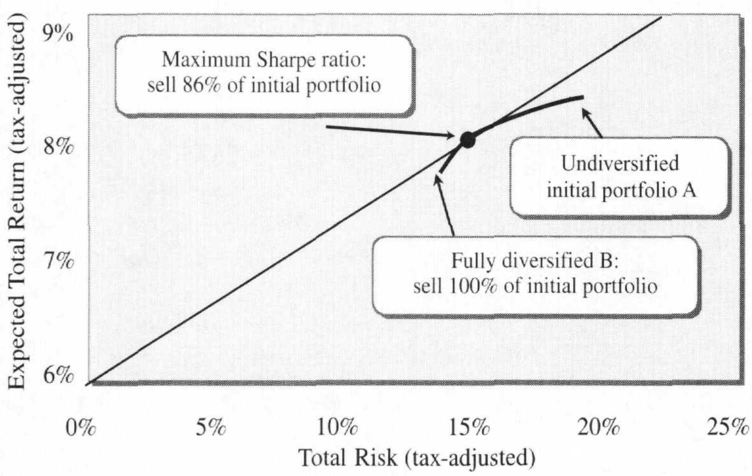


EXHIBIT 6A
SENSITIVITY TO INITIAL STOCK VOLATILITY
RISK-RETURN CURVES AT DIFFERENT LEVELS OF VOLATILITY

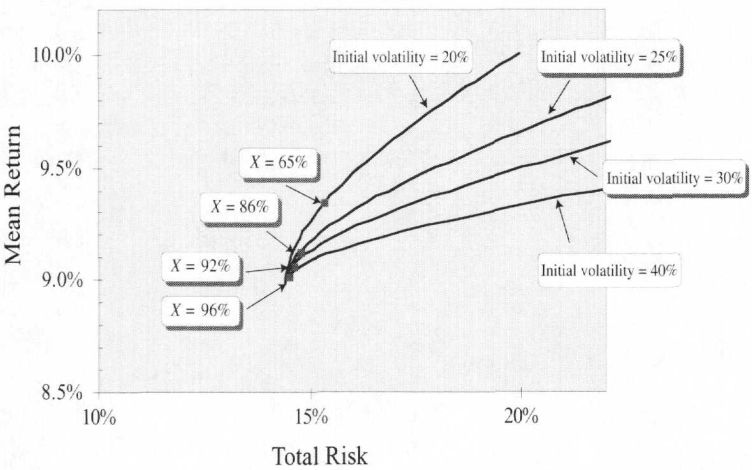
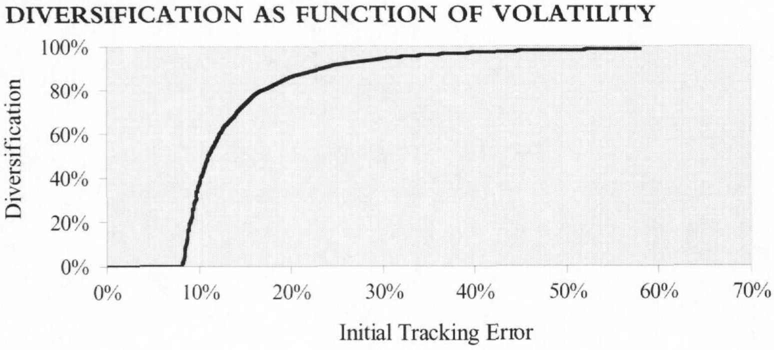


EXHIBIT 6B
SENSITIVITY TO INITIAL STOCK VOLATILITY
DIVERSIFICATION AS FUNCTION OF VOLATILITY



the maximum Sharpe ratio criterion recommends that 86% of the initial holding be sold. Greater diversification corresponds to a lower expected rate of return for the tax-deferred investor because it requires that higher taxes be paid initially. The tax-deferred investor must earn a lower annual rate of return in order to experience the same end-of-period performance as the actual investor.

What now is the sensitivity of this solution to changes in the base numerical parameters?

- Of key importance is the *risk of the initial holding*, σ_A. Exhibit 6 shows the risk-return trade-off and optimal diversification x for a range of risk levels. The more risky the stock A, the more it should be diversified. Many securities, such as those with volatility of more than 30%, should be almost completely diversified. There is less need to diversify low-volatility initial assets.
- The *horizon* is also important to the diversification decision, as shown in Exhibit 7. The longer the horizon, the more important risk becomes, and the more it should be diversified. For high-volatility initial holdings, the decision is not very sensitive to the horizon, and we recommend diversifying most of the asset. For low-volatility initial holdings, the horizon is more important.
- While the *cost basis* C_0 is important, too, its effect is straightforward. If the initial asset has a cost basis higher than zero, then diversification is cheaper. Thus, the diversification x increases with the cost basis, as shown in Exhibit 8. The relationship between cost basis and degree of diversification is close to linear.
- When the initial asset has a higher *expected return*, i.e., $\mu_A = \mu_B + \alpha$ with $\alpha > 0$, we would expect that the recommended diversification x decrease with the excess return α because a higher expected return makes the initial asset more valuable as compared to

EXHIBIT 7
SENSITIVITY TO HORIZON
DIVERSIFICATION AS FUNCTION OF HORIZON

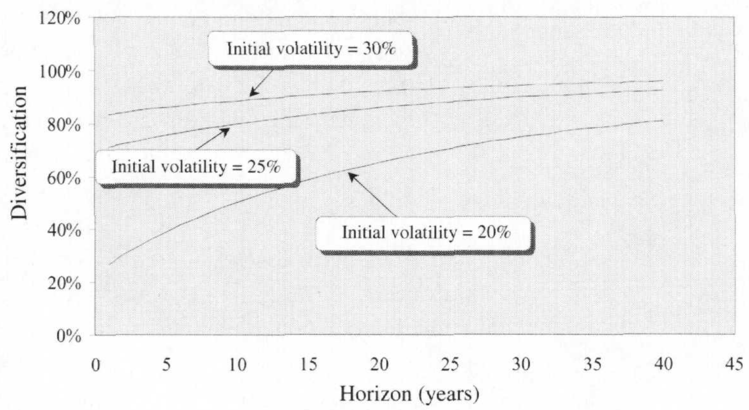


EXHIBIT 8
SENSITIVITY TO COST BASIS
DIVERSIFICATION AS FUNCTION OF COST BASIS

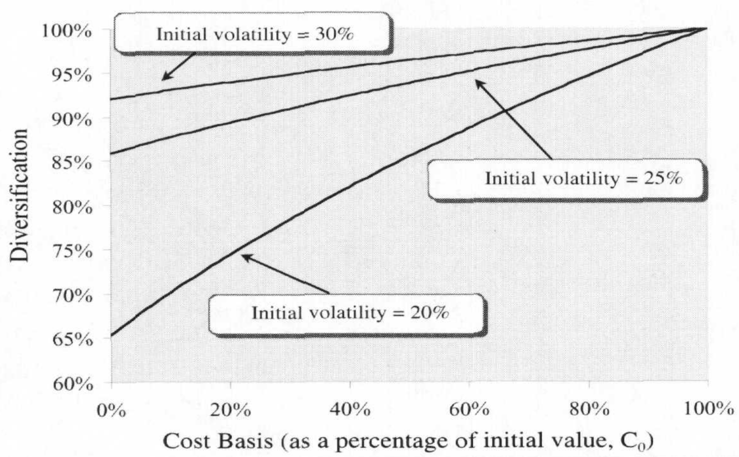
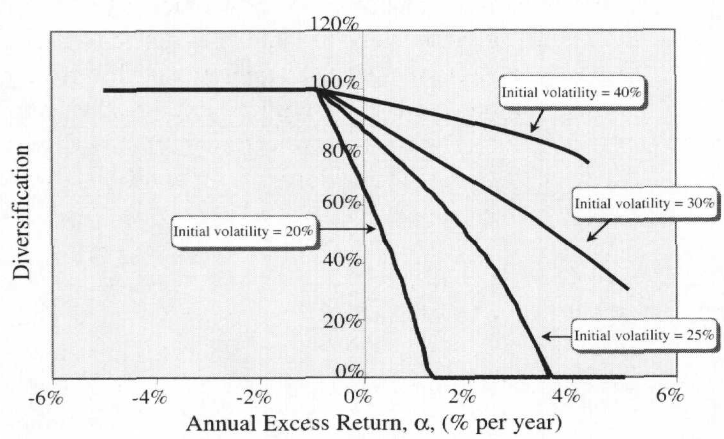


EXHIBIT 9
SENSITIVITY TO EXCESS RETURN
DIVERSIFICATION AS FUNCTION OF EXCESS RETURN



the benchmark. Exhibit 9 shows this. For example, if σ_A = 25% and α > 3.5% per year for 20 years, no diversification would be recommended. If σ_A = 30% and α = 2% per year for 20 years, the model would recommend diversifying about 70% of the holding.

- The effect of a lower *risk-free rate* r_f is to increase the level of diversification, as is clear from the convexity of the curve in Exhibit 5.
- We have assumed liquidation and the payment of capital gains taxes at the horizon. If the investor receives a step-up in basis at this time, diversification is more costly; the investor can avoid paying capital gains taxes by retaining the single stock, but is taxed for diversifying.

It is interesting to compare the model's recommended level of diversification in the step-up case with the liquidation case. As expected, for any given set of parameters our model always suggests less diversification in the step-up case. Differences are most pronounced for short horizons, as shown in Exhibit 10. The intuition is simple; if the horizon is short, the risk in holding the stock is relatively low, while the tax impact of selling it is relatively high.

We find that for horizons longer than 20 years, differences are not very great, and the risk of holding the single stock quickly overwhelms the tax benefit of retention. Thus, over long investment horizons, it is particularly unwise to incur the risk of concentration, even in the step-up case.

A RELATED PROBLEM: REDUCING TRACKING ERROR

A related problem concerns the holding of a low cost basis initial portfolio that is only partially diversified, and how it tracks a specified benchmark. The suitable measure of risk in this case is *tracking error*, rather than total standard deviation.[7]

EXHIBIT 10
SENSITIVITY TO HORIZON WITH AND WITHOUT COST BASIS STEP-UP
DIVERSIFICATION AS FUNCTION OF HORIZON
INITIAL VOLATILITY $\sigma_A = 25\%$

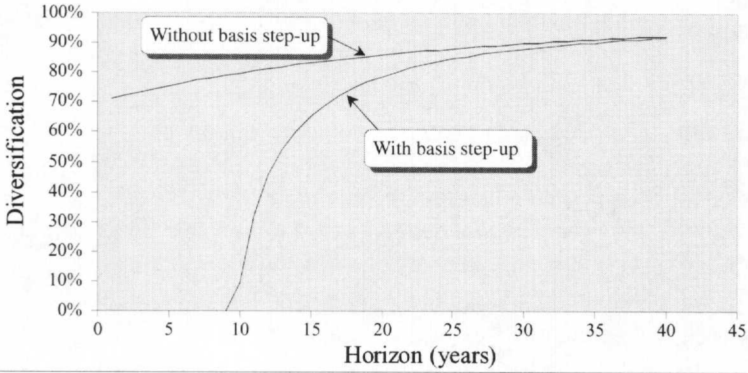
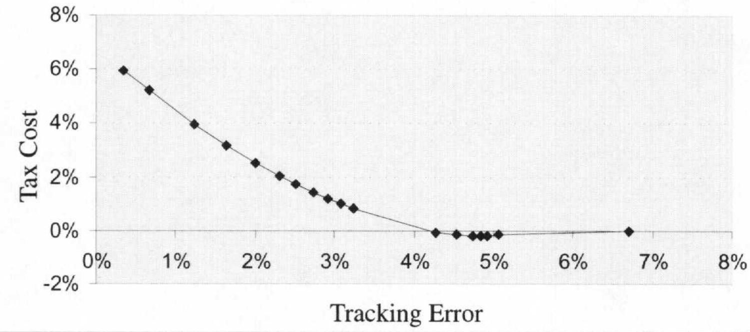


EXHIBIT 11
EMPIRICAL TRACKING ERROR VERSUS TAX COST



The investor reduces tracking error by selling first the tax lots that realize few taxes but that also provide the best opportunity for diversification. In general, the tax cost increases as the portfolio is squeezed down to track the benchmark. Mean-variance optimization can be used to minimize the tax cost and to provide a trade-off between tax cost and tracking error.

Exhibit 11 shows an empirical plot of the tax cost versus tracking error for a portfolio with an initial tracking error of 6.8%. A few initial tax lots have unrealized losses, and these allow the tracking error of the portfolio to be reduced to 4.2% before any net taxes are realized. Thereafter, the tax cost increases as tracking error decreases.

In this example, which point on the curve is best? Once again, by defining a tax-deferred equivalent investor, we can develop a solution. In this case, the similar tax-deferred problem is that faced by an investor who is seeking an active portfolio manager, trading off tracking error σ for excess return α. Such an investor typically seeks either an information ratio that is "high enough" (where the information ratio is measured by the slope α/σ) or maximization of utility $\alpha - \lambda\sigma^2$ for some given λ, the investor's risk preference.

Our analogous approach works for the tax management decision described here. The choice of λ is somewhat different from that described, for example, by Grinold and Kahn [1995] because when we compare α values at different diversification levels, the differences are not due to *uncertain* estimated portfolio performance, but instead come from *known* initial taxes paid. By reviewing the choices made by large numbers of investors, it would be possible to identify values of λ that are implicitly revealed by their preferences.

SUMMARY AND CONCLUSIONS

We have introduced a framework for trading off risk and return when diversifying low-basis taxable holdings. In the case of a risky *single asset*, we aim to reduce the total risk (standard deviation) of the asset. In the case of an initial *portfolio* that seeks to track a specified benchmark, we aim to reduce tracking error to an optimal level. In each case, we weigh the risk improvement against its tax cost.

When the initial asset has substantially more risk than the benchmark, our results recommend near-complete diversification, despite a high initial tax cost. If the initial asset's total risk is not much higher than that of the benchmark, the approach recommends less diversification, because the benefits do not cover the marginal tax cost. Sensitivity analysis reveals that greater diversification is needed: with greater initial asset volatility, with longer investment horizon, with a lower expected return of the initial asset, with a higher cost basis, and with a lower risk-free rate. Less diversification is needed when the investor receives a step-up in basis at the horizon.

Our approach has been to formulate a particularly simple decision problem. We have considered a single fixed-horizon investment, with only two possible extreme choices for portfolio formation. The formulation can be generalized in many pragmatically useful directions. One could:

- Include dividend yields, which affects the analysis because of the high rate of dividend taxation.
- Consider how an uncertain horizon affects decision-making.

Investors with large low-basis concentrated holdings are often reluctant to embrace our model's high diversification recommendations. For such investors, other pragmatic extensions are interesting. They might:

- Seek to compromise by staging the diversification over time; exploit tax-managed methods as in Stein and Narasimhan [1999] in managing the diversified slice to reduce the tax burden.
- Instead of investing in a diversified benchmark index, invest the liquidated asset in a portfolio that will "complete" the remaining undiversified holdings. That is, seek a portfolio that will have low (or ideally negative) correlation with the initial holdings.

In practice, investors may also be able to obtain additional flexibility with derivative securities, exchange funds, or other investment vehicles.

While we have focused on a particular and simplified analytical problem, our solution method can be quite generally applied to other portfolio decisions in the presence of taxes. In essence, our method translates a taxable problem into a tax-deferred equivalent problem based on annual mean and standard deviation. Instead of maximizing the Sharpe ratio or tracking error utility, one could choose the portfolio with maximum tax-deferred growth rate (please see endnote 3 for a discussion on growth rate), or use any other criterion for portfolio choice based on yearly mean and standard deviation. One can extend the concept of a matched tax-deferred investor to provide analytic intuitive solutions to a wide range of more complex situations.

APPENDIX
TECHNICAL DETAILS

We outline here the assumptions we use in defining the tax-deferred investor, and we then identify the future cash flows for both the taxable and the tax-deferred investor. Using these, we derive closed-form expressions for the tax-adjusted yearly expected rate of return and standard deviation of return. Finally, we show how these expressions must be modified for the case in which the investor receives a step-up in basis at maturity.

We assume that the actual investor initially holds a portfolio with initial market value W_0 and initial cost basis C_0W_0 so that C_0 represents the initial cost basis as a fraction between 0 and 1. This portfolio is assumed to grow at a random realized rate of return $A_i - 1$ in year i, so that the total horizon before-tax rate of return over n years is $\prod_{i=1}^{n} A_i - 1$.

The investor is considering selling a fraction x (between 0 and 1) of the initial portfolio and paying taxes of $\tau x W_0(1 - C_x)$ at rate τ on the proceeds xW_0 less the cost basis xC_xW_0 on shares sold. (Note that this formulation allows high cost basis shares to be chosen for sale.) The after-tax proceeds $xW_0[1 - \tau(1 - C_x)]$ are used to purchase shares of a benchmark portfolio with random realized rate of return $B_i - 1$ in year i. The resulting partially diversified portfolio now has cost basis $(C_0 - xC_x)W_0$ in the initial assets and full cost basis $xW_0[1 - \tau(1 - C_x)]$ in the newly purchased benchmark portfolio.

When the portfolio is liquidated after n years, the investor receives the compounded amount

$$(1-x)W_0\prod_{i=1}^{n}A_i + xW_0\left[1-\tau\left(1-C_x\right)\right]\prod_{i=1}^{n}B_i \tag{A-1}$$

and pays tax of

$$\tau W_0\left[(1-x)\prod_{i=1}^{n}A_i - \left(C_0 - xC_x\right)\right] + \tau x W_0\left[1-\tau\left(1-C_x\right)\right]\left(\prod_{i=1}^{n}B_i - 1\right) \tag{A-2}$$

resulting in a total after-tax compounded horizon rate of return equal to

$$r_x = (1-t)(1-x)\prod_{i=1}^{n}A_i + x(1-t)\left[1-t\left(1-C_x\right)\right]\times \prod_{i=1}^{n}B_i + tC_0 + xt(1-t)(1-C_x) - 1 \tag{A-3}$$

In order to replicate the investment performance of this actual investor, we seek to construct a tax-deferred investor (who does not pay tax initially, but whose after-tax end-of-horizon investment performance is identical to that of the actual investor) for each choice of x. Such a tax-deferred investor initially holds a portfolio with the same initial market value W_0 and initial cost basis C_0W_0 as the actual investor, but that returns $A^*_i - 1$ in year i.

The tax-deferred investor will sell a fraction x^* of the initial portfolio given by

$$x^* = x\left[1-\tau\left(1-C_x\right)\right] / \left[1-\tau x\left(1-C_x\right)\right] \tag{A-4}$$

where x^* is chosen so that the risk-return exposures of the actual and tax-deferred investors are equal (that is, x^* is set equal to the ratio, for the actual investor, of the after-tax dollar amount in B divided by total postdiversification portfolio value). The choice of x^* adjusts for the fact that the tax-deferred investor can invest the entire proceeds, $x^* W_0$, in the tax-deferred benchmark, which returns $B^*_i - 1$ in year i. Note that the probability distribution of (A^*, B^*) may depend on x. The tax-deferred investor retains the initial cost basis of $C_0 W_0$.

When the tax-deferred portfolio is liquidated after n years, the tax-deferred investor receives the compounded amount

$$W_0\left[(1-x^*)\prod_{i=1}^n A_i^* + x^* \prod_{i=1}^n B_i^*\right] \tag{A-5}$$

and pays tax at the end of the time horizon in the amount of

$$\tau W_0\left[(1-x^*)\prod_{i=1}^n A_i^* + x^* \prod_{i=1}^n B_i^* - C_0\right] \tag{A-6}$$

resulting in a total after-tax compounded horizon rate of return equal to

$$r_x^* = (1-\tau)(1-x^*)\prod_{i=1}^n A_i^* + (1-\tau)x^* \prod_{i=1}^n B_i^* + \tau C_0 - 1 \tag{A-7}$$

In the fully diversified case ($x = x^* = 1$), we have total rates of return r_B and r^*_B for the actual and tax-deferred investors:

$$r_B = (1-\tau)\left[1-\tau\left(1-C_0\right)\right]\prod_{i=1}^n B_i + \tau(1-\tau) + \tau^2 C_0 - 1$$
$$r_B^* = (1-\tau)\prod_{i=1}^n B_i^* + \tau C_0 - 1 \tag{A-8}$$

Uncertainty is specified as follows. The distribution of (A_i, B_i) is joint lognormal, independent for different years, with means (μ_A, μ_B), standard deviations (σ_A, σ_B), and instantaneous beta β. Similarly, the distribution of (A^*_i, B^*_i) is joint lognormal, independent for different years, with means (μ_{A*}, μ_{B*}), standard deviations $(\sigma_{A*}, \sigma_{B*})$, and instantaneous beta β^* (which may differ from β due to initial taxes).

To find the joint distribution (specified by μ_{A*}, μ_{B*}, σ_{A*}, σ_{B*}, and β^*) for the tax-deferred investor, moment conditions are imposed in order to make the joint distribution of after-tax compounded horizon rates of return (of the partially diversified portfolio and the benchmark) nearly identical for the actual and the tax-deferred investors at this particular value for x. That is, the joint probability distribution of (r_x, r_B) is closely matched to that of (r^*_x, r^*_B) using the five moment conditions:

$$E(r_x) = E(r^*_x)$$
$$E(r_B) = E(r^*_B) \tag{A-9}$$

$$\sigma_{r_x} = \sigma_{r^*_x}$$
$$\sigma_{r_B} = \sigma_{r^*_B} \tag{A-10}$$

$$Cov(r_x, r_B) = Cov(r^*_x, r^*_B) \tag{A-11}$$

The yearly expected rates of return for the tax-deferred investor can then be shown to be given by

$$\mu_{B^*} = \left\{\left[1-\tau\left(1-C_0\right)\right]\left(1+\mu_B\right)^n + \tau\left(1-C_0\right)\right\}^{1/n} - 1 \tag{A-12}$$

$$\mu_{A^*} = \left(\frac{(1-x)\left(1+\mu_A\right)^n + x\left[1-\tau\left(1-C_x\right)\right]\left(1+\mu_B\right)^n - x^*\left(1+\mu_{B^*}\right)^n + \tau x\left(1-C_x\right)}{1-x^*}\right)^{1/n} - 1 \tag{A-13}$$

Define $\nu_A = 1 + \mu_A$, $\theta^2_A = \sigma^2_A + \nu^2_A$, and similarly for A^*, B, and B^* to reduce the complexity of the equations to follow, and also define $\delta = \nu_A \nu_b(\theta^2_B/\nu^2_B)^\beta$ and similarly $\delta^* = \nu_{A*}\nu_{b*}(\theta^2_{B*}/\nu^2_{B*})^{\beta*}$. The yearly standard deviations and systematic risk β^* for the tax-deferred investor are given by

$$\theta^2_{B^*} = \left\{\begin{matrix}\left[1-\tau\left(1-C_0\right)\right]^2\left[\theta_B^{2n} + \dfrac{2\tau}{1-\tau}\nu_B^n + \dfrac{\tau^2}{(1-\tau)^2}\right] - \\ \dfrac{2\tau C_0}{1-\tau}\nu^n_{B*} - \left(\dfrac{\tau C_0}{1-\tau}\right)^2\end{matrix}\right\}^{1/n} \tag{A-14}$$

$$\sigma_{B^*} = \sqrt{\theta^2_{B^*} - \left(1+\mu_{B^*}\right)^2} \tag{A-15}$$

$$\begin{aligned}\delta^* = \Bigg\{ \frac{1}{1-x^*}\Bigg(& \left[\frac{\tau C_0}{1-\tau} + \tau x\left(1-C_x\right)\right]\left\{\left(\tau + \frac{\tau^2 C_0}{1-\tau}\right) + \left[1-\tau\left(1-C_0\right)\right]\nu_B^n\right\} \\ & + \left(\tau + \frac{\tau^2 C_0}{1-\tau}\right)\left\{(1-x)\nu_A^n + x\left[1-\tau\left(1-C_x\right)\right]\nu_B^n\right\} \\ & + x\left[1-\tau\left(1-C_x\right)\right]\left[1-\tau\left(1-C_0\right)\right]\theta_B^{2n} + (1-x)\left[1-\tau\left(1-C_0\right)\right]\delta^n \\ & - \left(\frac{\tau C_0}{1-\tau}\right)^2 - \frac{\tau C_0}{1-\tau}\left[\left(1-x^*\right)\nu^n_{A^*} + \left(1+x^*\right)\nu^n_{B^*}\right] - x^*\theta^{2n}_{B^*}\Bigg)\Bigg\}^{1/n}\end{aligned} \tag{A-16}$$

$$\beta^* = \frac{\ln\left(\frac{\delta^*}{(1+\mu_{A^*})(1+\mu_{B^*})}\right)}{\ln\left(\frac{\sigma_{B^*}^2 + (1+\mu_{B^*})^2}{(1+\mu_{B^*})^2}\right)} \tag{A-17}$$

$$\theta_{A^*}^2 = \left\{ \frac{1}{(1-x^*)^2} \left(\left[\frac{\tau C_0}{1-\tau} + \tau x(1-C_x)\right] \left\{ \left[\frac{\tau C_0}{1-\tau} + \tau x(1-C_x)\right] + 2(1-x)v_A^n + 2x[1-\tau(1-C_x)]v_B^n \right\} + (1-x)^2\theta_A^{2n} + x^2[1-\tau(1-C_x)]^2\theta_B^{2n} + 2x(1-x)[1-\tau(1-C_x)]\delta^n - \frac{\tau C_0}{1-\tau}\left[\frac{\tau C_0}{1-\tau} + 2(1-x^*)v_{A^*}^n + 2x^* v_{B^*}^n\right] - (x^*)^2\theta_{B^*}^{2n} - 2x^*(1-x^*)(\delta^*)^n \right) \right\}^{1/n} \tag{A-18}$$

$$\sigma_{A^*} = \sqrt{\theta_{A^*}^2 - (1+\mu_{A^*})^2} \tag{A-19}$$

The tax-adjusted yearly expected rate of return and standard deviation may now be computed using the values derived above:

$$\begin{aligned} \mu_x &= E\left[(1-x^*)A_1^* + x^* B_1^*\right] \\ &= (1-x^*)\mu_{A^*} + x^*\mu_{B^*} \end{aligned} \tag{A-20}$$

$$\begin{aligned} \sigma_x &= \sqrt{\mathrm{Var}\left[(1-x^*)A_1^* + x^* B_1^*\right]} \\ &= \sqrt{(1-x^*)^2\theta_{A^*}^2 + (x^*)^2\theta_{B^*}^2 + 2x^*(1-x^*)\delta^* - \left[(1-x^*)v_{A^*} + x^* v_{B^*}\right]^2} \end{aligned} \tag{A-21}$$

If the investor receives a step-up in basis at maturity, then the partially diversified investor keeps the compounded amount from Equation (A-1) without paying the tax of Equation (A-2), resulting in a total after-tax compounded horizon rate of return equal to

$$r_x = (1-x)\Pi_{i=1}^n A_i + x[1-\tau(1-C_x)]\Pi_{i=1}^n B_i - 1 \tag{A-22}$$

in place of Equation (A-3). We keep the definition of x^* from Equation (A-4) unchanged. For the stepped-up tax-deferred investor, in place of Equation (A-7) we find a total after-tax compounded horizon rate of return equal to

$$r_x^* = (1-x^*)\Pi_{i=1}^n A_i^* + x^*\Pi_{i=1}^n B_i^* - 1 \tag{A-23}$$

In the fully diversified case ($x = x^* = 1$), we have total rates of return r_B and r_B^* for the stepped-up actual and tax-deferred investors [in place of Equation (A-8)]:

$$\begin{aligned} r_B &= [1-\tau(1-C_0)]\Pi_{i=1}^n B_i - 1 \\ r_B^* &= \Pi_{i=1}^n B_i^* - 1 \end{aligned} \tag{A-24}$$

We use the same forms for the joint distributions of (A_i, B_i) and (A_i^*, B_i^*) as before, and use the same five moment conditions [Equations (A-9) through (A-11)]. The yearly expected rates of return for the stepped-up tax-deferred investor can then be shown to be given [in place of Equations (A-12) and (A-13)] by

$$\mu_{B^*} = (1+\mu_B)[1-\tau(1-C_0)]^{1/n} - 1 \tag{A-25}$$

$$\mu_{A^*} = \left(\frac{(1-x)(1+\mu_A)^n + x[1-\tau(1-C_x)](1+\mu_B)^n - x^*(1+\mu_{B^*})^n}{1-x^*}\right)^{1/n} - 1 \tag{A-26}$$

Using definitions of v, θ, and δ as before, the yearly standard deviations for the stepped-up tax-deferred investor are then given [in place of Equations (A-14) through (A-19)] by

$$\theta_{B^*} = \theta_B[1-\tau(1-C_0)]^{1/n} \tag{A-27}$$

$$\sigma_{B^*} = \sqrt{\theta_{B^*}^2 - (1+\mu_{B^*})^2} \tag{A-28}$$

$$\delta^* = \left(\frac{x[1-\tau(1-C_x)][1-\tau(1-C_0)]\theta_B^{2n} - x^*\theta_{B^*}^{2n} + (1-x)[1-\tau(1-C_0)]\delta^n}{1-x^*}\right)^{1/n} \tag{A-29}$$

$$\beta^* = \frac{\ln\left(\frac{\delta^*}{(1+\mu_{A^*})(1+\mu_{B^*})}\right)}{\ln\left(\frac{\sigma_{B^*}^2 + (1+\mu_{B^*})^2}{(1+\mu_{B^*})^2}\right)} \tag{A-30}$$

$$\theta^2_{A^*} = \left(\frac{(1-x)^2\theta_A^{2n} + x^2\left[1-\tau(1-C_x)\right]^2\theta_B^{2n} - \left(x^*\right)^2\theta_{B^*}^{2n} + 2x(1-x)\left[1-\tau(1-C_x)\right]\delta^n - 2x^*\left(1-x^*\right)\left(\delta^*\right)^n}{\left(1-x^*\right)^2} \right)^{1/n} \quad \text{(A-31)}$$

$$\sigma_{A^*} = \sqrt{\theta^2_{A^*} - \left(1+\mu_{A^*}\right)^2} \quad \text{(A-32)}$$

With these modifications, the tax-adjusted yearly expected rate of return and standard deviation in the case of stepped-up basis may now be computed as before, using Equations (A-20) and (A-21).

ENDNOTES

At the time of this writing, Charles Appeadu was a portfolio manager at Parametric Portfolio Associates.

[1]We use the term *investment performance* to refer to the after-tax end-of-horizon cash flow probability distribution.

[2]At an inflation rate of about 3.5% for 20 years, the initial $1 million value doubles to $2 million in 20 years.

[3]This phenomenon is due to the fact that the long-term *growth rate* (see, e.g., Fernholz and Shay [1982]) is *less* than the yearly *expected rate of return* due to a risk penalty (equal to half the variance in the case of a lognormal distribution). Some intuition into this paradox is provided by the simple example of gaining 20% or losing 20% with probability one-in-two. The expected rate of return is zero, even over the long run. You would have to be very lucky not to lose money over the long run, however, because the particular sequence "gain 20%, then lose 20%" reduces wealth by 4% [computed as $(1 + 0.2)(1 - 0.2) - 1$] or about 2% each time the example is played (when there are exactly equal numbers of ups and downs). This 2% reduction is indeed half the variance since $0.2^2/2 = 2\%$. Curiously, while the compounded expected rate of return is equal to the expected compounded rate of return, over the long run this rate becomes nearly impossible to attain due to the risk penalty.

[4]Brunel [1998] emphasizes that taxable investors should use caution when using traditional efficient frontier tools directly.

[5]For simplicity, we assume no transaction costs. This assumption is reasonable when transaction costs are small compared to tax costs.

[6]The analysis may also be generalized to the case of an investor who does not liquidate, but who receives a step-up in cost basis at death.

[7]Tracking error is the standard deviation of the annual difference between the return of the portfolio and that of the benchmark.

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